Normal families

Let X, Y be two Riemann surfaces, and $\mathcal{F} \subset Hol(X,Y)$ a family of holomorphic maps

Suppose for now that Y is compact.

Def: A family $\mathcal{F} \subset Hol(X,Y)$ (Y compact) is called normal if $\overline{\mathcal{F}} \subset Hol(X,Y)$ is compact.

Equivalently, if every infinite sequence $(f_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ admits a subsequence f_{n_k} converging uniformly on compacts of X to a function $g: X \to Y$ ($g \in Hol(X,Y)$, not necessarily in $\mathcal{F}$).

Example: $f: \hat{\mathbb{C}} \to \hat{\mathbb{C}}$, $\mathcal{F} = \{ f^n \mid n \in \mathbb{N} \}$ $f(z) = z^2$.

Consider $\mathbb{D} \subset \hat{\mathbb{C}}$, and $\mathcal{F}_0 = \{ f^n|_{\mathbb{D}} : n \in \mathbb{N} \}$, $X = \mathbb{D}$, $Y = \hat{\mathbb{C}}$.

In this case for any sequence $(f^{n_j}|_{\mathbb{D}})_{j\in\mathbb{N}}$ we may find a subsequence $f^{n_{j_k}}$ (setting $j_0 = 0$, j_{k+1} such that $n_{j_{k+1}} > n_{j_k}$) which converges uniformly on compacts of $\mathbb{D}$ to the constant function 0.

Analogously, for $\mathbb{D}_\infty = \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and $\mathcal{F}_\infty = \{ f^n|_{\mathbb{D}_\infty}, n \in \mathbb{N} \}$, we have that any sequence admits a converging subsequence (to the constant function ∞)

Notice that $\forall p \in \partial\mathbb{D}$, $\forall \varepsilon > 0$, the family $\mathcal{F}_{p,\varepsilon} = \{ f^n|_{\mathbb{D}(p,\varepsilon)} : n \in \mathbb{N} \}$ is not normal, since any possible limit function $g: \mathbb{D}(p,\varepsilon) \to \hat{\mathbb{C}}$ would take the value 0 on $\mathbb{D}(p,\varepsilon) \cap \mathbb{D}$, and ∞ on $\mathbb{D}(p,\varepsilon) \cap \mathbb{D}_\infty$, Hence g is not continuous: a contradiction.

Notice that if we consider $f:\mathbb{C}\to\mathbb{C}$ and $\mathcal{F}=\{f^n|_{\mathbb{C}\setminus\overline{\mathbb{D}}}\ n\in\mathbb{N}\}$, further family $\overline{\mathcal{F}}$ is not compact. ~~To deal~~

To deal with non-compact targets, we need the following definitions:

Def Let Y be a (possibly non-compact) Riemann surface.

A sequence of points $(y_n)_{n\in\mathbb{N}}$ in Y "diverges from Y" if $\forall K\subseteq Y$ compact, $y_n\notin K$ $\forall n>>0$. (i.e. $\exists N_K\in\mathbb{N}$, $y_n\notin K$ $\forall n\geq N_K$)

Similarly, a sequence of maps $f_n: X\to Y$ "diverges locally uniformly from Y" if $\forall K_x\subseteq X$, $K_y\subseteq Y$ compact sets, we have that $f_n(K_x)\cap K_y=\emptyset$ $\forall n>>0$. ($\exists N=N(K,K')$ ...)

Example: the sequence $f^n|_{\mathbb{C}\setminus\overline{\mathbb{D}}}$, where $f(z)=z^2$, diverges uniformly on from $\mathbb{C}$:

Any $K_x\subset\mathbb{C}\setminus\overline{\mathbb{D}}$ is contained in $\{|z|\geq r\}$ for some $r>1$,

Any $K_y\subset\mathbb{C}$ is contained in $\{|z|\leq R\}$ for some $R>0$.

Then $f^n(K_x)\cap K_y=\emptyset$ as far as $r^{2^n}>R$.

Rem: of course divergence from Y may occur only when Y is not compact.

Def: Let X, Y be two Riemann surfaces and $\mathcal{F}\subset \mathrm{Hol}(X,Y)$ be a family of holomorphic maps.

Then $\mathcal{F}$ is called normal if $\forall (f_n)\subset\mathcal{F}$ infinite sequence admits

~~other~~ a subsequence that either converges locally uniformly, or that diverges locally uniformly from Y. (3.11)

Corollary (of hyperbolic HC compactness). If X, Y are hyperbolic, then every family $\mathcal{F}$ of holomorphic maps $X \to Y$ is normal

Proof: Pick any $x_0 \in X$.

If the set $\{f(x_0) \mid f \in \mathcal{F}\}$ lies in a compact $K_Y \subset Y$, then by the Theorem of H.C., $\mathcal{F} \subset \mathcal{F}' = \{f \in Hol(X,Y) \mid f(x_0) \in K_Y\}$. $\mathcal{F}'$ is compact, and hence so is $\overline{\mathcal{F}}$ (closed in compact is compact)

Assume there is no such compact. Take any sequence $(f_n) \subset \mathcal{F}$.

Again, if $\exists K_Y \subset Y$ compact containing $\{f_n(x_0) \mid n \in \mathbb{N}\}$, then $\overline{\{f_n\}}$ is compact and there exists a subsequence converging uniformly on compact sets. Assume this is not the case. Fix $y_0 \in Y$. Then for any $k \in \mathbb{N}$, $\exists n_k$ so that $\rho_Y(f_{n_k}(x_0), y_0) > k$ $(*)$

(Being $\overline{B_{\rho_Y}(y_0, k)}$ compact). (Hence $\rho_Y(f_{n_k}(x_0), y_0) \to +\infty$.)

Since $(B_{\rho_Y}(y_0, k))_k$ is an increasing sequence of (compact) sets, we may assume that the sequence n_k is increasing

For any $K_X \subset X$ compact, we have that $\forall x \in K_X$ $\rho_Y(f_{n_k}(x), f_{n_k}(x_0)) \leq \rho_X(x, x_0) \leq R$

For some R depending on K_X. Then $\rho_Y(f_{n_k}(x), y_0) \geq k - R$ for any k.

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In particular, $\forall K_Y \subset Y$, it is contained in $B_{g_Y}(y_0, S)$ for some S, and $f_{n_k}(K_X) \cap K_Y = \emptyset$ as far as $k - R > S$, i.e., (f_{n_k}) diverges uniformly from Y. ◻

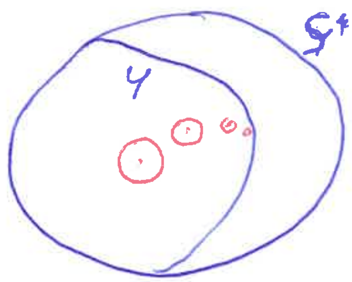
Let $F \subset Hol(X,Y)$ be a family of holomorphic maps. We can study the normality on any connected open subset $U \subset X$, either by studying the family $F|_U \subset Hol(U,Y)$, where $F|_U = \{ f|_U : U \to Y \mid f \in F\}$, or by studying local convergence / divergence from Y of F on compacts of U.

We want to understand now what happens if $Y \subset S$, and we consider F as a family on $Hol(X, S)$.

Theorem (Poincaré metric near the boundary).

Let $Y \subset S$ be two Riemann surfaces, with Y hyperbolic

Let $(p_j)_{j \in \mathbb{N}}$ be a sequence in Y that converges (w.r.t. the topology in S) to a boundary point $p_\infty \in \partial Y \subset S$.



Then $\forall r > 0$ fixed, $B_{g_Y}(p_j, r)$ converges uniformly to p_∞ as $j \to +\infty$.

More generally, if $Y \subset\subset S$, we can take any sequence (p_n) diverging from Y

~~with limit points~~

(The convergence is uniform on compact sets of S (?))

Proof:

Step 1: $\forall K \subset Y$ compact, then $B_{\rho_Y}(p_j, r) \cap K = \emptyset$ for $j \gg 0$.

In fact $K \subset B_{\rho_Y}(y_K, r_K)$ for some $y_K \in Y$, $r_K > 0$.

Since $p_j \to p_\infty \in \partial Y$, $\rho_Y(y_K, p_j) \to +\infty$. In particular, $\rho_Y(y_K, p_j) > r + r_K$

for $j \gg 0$, and $K \cap B_{\rho_Y}(p_j, r) = \emptyset$.

Step 2: Set $N_r = B_{\rho_{\mathbb{D}}}(0, r) \subset \mathbb{D}$. Fix a universal covering $pr: \mathbb{D} \to Y$.

and ~~compose~~ compose it with an element ϕ_j of $Aut(\mathbb{D})$ so that $f_j = pr \circ \phi_j$

is a covering map sending 0 to $f_j(0) = p_j$.

Then $B_{\rho_Y}(p_j, r) = f_j(N_r)$.

Step 3: $\forall K \subset Y$ compact and large enough, $S \setminus K$ is a hyperbolic Riemann surface (If $S \simeq \hat{\mathbb{C}}$ is easy; if $S \setminus K$ is covered by $\mathbb{C}$ just eliminate a small ball B, which will be covered by $\mathbb{C} \setminus pr^{-1}(B)$, which is hyperbolic).

By step 1 and construction in step 2, $f_j(N_r) \cap K = \emptyset$ for j big enough, and by the corollary $\{ f_j|_{N_r} : N_r \to S \setminus K,\ j \gg 0 \}$ is a normal family. Since $p_j \to p_\infty \in S$, all the p_j belong to some compact $K_S \subset S$ (*)

Then we can find a subsequence f_{j_k} that converges locally uniformly to some $f: N_r \to S \setminus K$ (since we cannot have divergence from $S \setminus K$).

Rem: the convergence is ~~also uniform~~ uniform on $\overline{N_r}$ if we apply the same argument to N_{r+1}.

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Step 4: f is a constant map $(f \equiv p_\infty)$

If not, f would be open, and $f(N_r)$ would be an open subset of $S \setminus K$. In particular, it would intersect Y. But this is a contradiction, since Y can be exausted by compact sets $(H_i)_i$, and ~~we~~ we have seen that $\forall i$ $f_j(N_2) \cap H_i = \emptyset$ for j big enough.

Step 5. We showed that $B_{g_Y}(p_{j_k}, r) = f_{j_k}(N_2) \to p_\infty$ uniformly.

We must show that the same happens to the whole sequence.

Choose any distance d_S on S (compatible with its topology), i.e. $\left.\begin{matrix} g_S \\ \text{flat} \\ \text{spheric} \end{matrix}\right\}$ if S is $\left\{\begin{matrix} \text{hyperbolic} \\ \text{parabolic} \\ \text{elliptic} \end{matrix}\right)$

And let $d_j = \mathrm{diam}(f_j(N_2))$.

Suppose by contradiction that $d_j \not\to 0$: $\exists \varepsilon > 0$, $\exists j_k$ subsequence so that $d_{j_k} > \varepsilon$ $\forall k$. By using the same argument as in step 4, we can find another subsequence h_k so that $f_{j_{h_k}} \to p_\infty$ uniformly, a contradiction with $d_{j_k} > \varepsilon$.

$\square$

Notice that all the arguments work as far as ~~the~~ $\{p_j\}$ is contained in a compact subset of S. In particular the argument works if $Y \subset\subset S$ (relatively compact) for any sequence p_j diverging from Y.

In this case we wouldn't have that $f_j(N_2) \to p_\infty$ (since (p_j) could not converge in S), but we would still have $\mathrm{diam}_{d_S}(f_j(N_2)) \to 0$.

We deduce the following theorem:

Theorem. Let X, Y, S be three Riemann surfaces, with $Y \subset S$.
A family $\mathcal{F} \subset \mathrm{Hol}(X,Y)$ is normal if and only if it is normal when considered as a family $\mathcal{F} \subset \mathrm{Hol}(X,S)$.

Proof: We must show that if a sequence $(f_n) \subset \mathcal{F}$ diverges locally uniformly from Y, but not from S, then there is a subsequence f_{n_k} that converges locally uniformly (on S).

We will show that we can find one that converges to a constant map $S \to p_\infty \in \partial Y \subset S$.

The condition correspond to the existence of $K_X \subset X$ and $K_S \subset S$ compact sets so that $f_{n_j}(K_X) \cap K_S \neq \emptyset$ $\forall j$, for a suitable n_j subsequence.

Up to passing to a subsequence, we can pick points x_j in K_X so that $f_{n_j}(x_j) \in K_S$ and $f_{n_j}(x_j) \xrightarrow[j\to\infty]{} \ell \in K_S$. Since f_n diverges locally uniformly from Y, $\ell \in \partial Y$.

First suppose that X and Y are hyperbolic.

because not expanding.

$K_X \subset B_{\rho_X}(\bar{x}, r)$ for some $\bar{x} \in X$, $r < +\infty$ $\Rightarrow$ $f_{n_j}(K_X) \subset B_{\rho_Y}(f_{n_j}(\bar{x}), r)$

$\Rightarrow f_{n_j}(K_X) \subset B_{\rho_Y}(x_j, 2r) \xrightarrow{\text{uniformly}} \ell$ (by the previous theorem)

Since we can take any $K \supset K_X$ compact and repeat the same argument, we deduce that $f_{n_j} : X \to S$ converges locally uniformly to the constant map $x \mapsto \ell$.

In the general case, we may repeat the previous argument for $K^c X_0 \subset\subset X$ hyperbolic neighborhood of K_x, ~~with compact closure~~ and replacing Y with $Y_0 = Y \setminus H$ for some compact H.

Since X_0 can be taken as large as wanted, we conclude □

As a corollary, we get:

Theorem (Montel): let X be any Riemann surface, and $\mathcal{F} \subset Hol(X, \hat{\mathbb{C}})$ that omits at least three values (i.e., $\exists\, a \neq b \neq c$ in $\hat{\mathbb{C}}$, $\mathcal{F} \subset Hol(X, \hat{\mathbb{C}} \setminus \{a,b,c\})$.) The $\mathcal{F}$ is a normal family

Proof: $\forall U \subset X$ hyperbolic, $\mathcal{F}|_U$ is normal, since $\hat{\mathbb{C}} \setminus \{a,b,c\}$ is an hyperbolic surface. Being normality a local property, ~~and~~ We can cover X by hyperbolic surfaces (discs), and $\mathcal{F}$ is normal. □.

Normality and Equicontinuity.

Definition. A family $\mathcal{F} \subset \mathcal{C}^0(X, Y)$ between two metric spaces is equicontinuous if $\forall \varepsilon > 0 \ \exists\, \delta \overset{=\delta(\varepsilon)}{>} 0$ such that

$\forall f \in \mathcal{F}, \ \forall x_0, x \in X, \ d_X(x, x_0) < \delta \Rightarrow d_Y(f(x), f(x_0)) < \varepsilon.$

$\mathcal{F}$ is locally equicontinuous if $\forall x_0 \in X \ \exists U_0 \subset X$ neighborhood of x_0 s.t. $\mathcal{F}|_U$ is equicontinuous (or equivalently, if $\mathcal{F}|_K$ is equicontinuous $\forall K \subset X$ compact set).

Theorem (Ascoli-Arzelà) (in our setting)

Let X, Y be two Riemann surfaces, Y compact (X, Y complete, X separable)

(More generally X, Y metric spaces), and $\mathcal{F} \subset Hol(X,Y)$ $(\mathcal{F} \subset \mathcal{C}^0(X,Y))$.

Then $\mathcal{F}$ is normal $\iff$ $\mathcal{F}$ is locally equicontinuous.

(If Y is not compact, we have that the following conditions are equivalent:

1) $\forall (f_n) \subset \mathcal{F}$ sequence admits a subsequence converging locally uniformly

2) a) $\mathcal{F}$ is locally equicontinuous, and

b) $\forall x \in X$ $\exists K_{Y,x} \subset Y$ compact such that $\{f(x) \mid f \in \mathcal{F}\} \subset K_{Y,x}$.)

Proof. ($1 \Rightarrow 2$ a) By contradiction, assume that $\mathcal{F}$ is normal, but not locally equicontinuous. Hence:

$\exists K_X \subset X$ compact, $\exists \varepsilon > 0$ s.t. $\forall \delta = \frac{1}{n} > 0$ $\exists f_n \in \mathcal{F}$, $\exists p_n, q_n \in K_X$ s.t.

$$d_X(p_n, q_n) < \frac{1}{n} \text{ and } d_Y(f_n(p_n), f_n(q_n)) > \varepsilon. \qquad (*)$$

Since $\mathcal{F}$ is normal, we can pick a subsequence (f_{n_k}) converging uniformly on compacts ~~(in particular on K)~~ towards a map $f_\infty : X \to Y$ holomorphic (suffices continuous).

Up to taking a subsequence, since K_X is compact, we may also assume that $p_{n_k} \to p_\infty \in K_X$ and $q_{n_k} \to q_\infty \in K_X$.

By $(*)$, we have $d_X(p_{n_k}, q_{n_k}) < \frac{1}{n_k} \to 0$, and $p_\infty = q_\infty$.

But $\varepsilon < d_Y(f_{n_k}(p_{n_k}), f_{n_k}(q_{n_k})) \xrightarrow{\text{continuity}} d_Y(f_\infty(p_\infty), f_\infty(q_\infty)) = 0$, contradiction.

$(1 \Rightarrow 2b)$ We show that $\overline{F(x)}$ is compact in Y.

Let (y_n) be any sequence in $\overline{F(x)}$. Hence $\forall n \in \mathbb{N}$, $\exists f_n \in F$ so that $d_Y(f_n(x_0), y_n) < \frac{1}{n}$

Since F satisfies 1, $\exists f_{n_k}$ subsequence converging locally uniformly to f_∞.

Then $y_{n_k} \to f_\infty(x)$ $\left(d_Y(y_{n_k}, f_\infty(x)) < \underbrace{d_Y(y_{n_k}, f_{n_k}(x))}_{\frac{1}{n_k} \to 0} + \underbrace{d(f_{n_k}(x), f_\infty(x))}_{\to 0} \right)$

$(2 \Rightarrow 1)$ Let $(f_n) \subset F$ be any sequence. Being X separable, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ (all distinct) so that $\{x_n\}_{n \in \mathbb{N}}$ is dense in X.

We proceed by diagonal construction:

Since $\{f_n(x_0)\} \subset F(x_0)$ is contained in a compact set, we may find a subsequence $(f_{n_{0,j}})$ so that $f_{n_{0,j}}(x_0) \to y_0 \in Y$.

Set $f_\infty(x_0) = y_0$. Call $Q_0 = \{n_{0,j} \mid j \in \mathbb{N}\}$.

Consider now $\{f_{n_{0,j}}(x_1)\}$. Again we can extract a subsequence (whose set of indices we call Q_1) so that $f_n(x_1) \xrightarrow[n \in Q_1]{n \to \infty} y_1 \in Y$.

$=: f_\infty(x_1)$.

By recursion, we construct a sequence of nested subsequences indicized by $Q_0 \supset Q_1 \supset \dots \supset Q_k \supset \dots$ so that $\lim_{\substack{n \to \infty \\ n \in Q_k}} f_n(x_k) = y_k =: f_\infty(x_k)$.

Let $\hat{Q}$ be the set of indices built by taking the 1st of Q_0, the second of Q_1, etc.

We have that $\lim_{\substack{n \to \infty \\ n \in \hat{Q}}} f_n(x_k) = f_\infty(x_k)$ $\forall k \in \mathbb{N}$.

We claim that $(f_n)_{n \in \hat{Q}}$ converges uniformly on compacts.

Fix $K = K_x \subset X$ compact.

Being $\mathcal{F}$ equicontinuous, $\forall \varepsilon > 0\ \exists \delta = \delta(\varepsilon) > 0$ s.t. $\forall f_n, n \in \hat{Q}, \forall x, x' \in K$,

$d_x(x, x') < \delta \Rightarrow d_Y(f_n(x), f_n(x')) < \varepsilon$.

Fix such ε. Since K is compact, there exist $\ldots > 0$ such that $K \subset \bigcup_{j=0} B_{d_x}(x_j, \delta)$.

In other words, $\forall x \in K\ \exists j \in \{0 \ldots\}$ s.t. $d_x(x, x_j) < \delta$.

Moreover since $f_n(x_j) \xrightarrow[n \in \hat{Q}]{} f_\infty(x_j)\ \forall j = 0 \ldots$ ~~such that~~ we have that $\forall \varepsilon > 0$

$\exists N = N(\varepsilon)$ such that $\forall n, m \geq N, m, n \in Q$, we have $d_Y(f_n(x_j), f_m(x_j)) < \varepsilon\ \forall j = 0 \ldots M$.

We deduce that $\forall x \in K$, $\forall n, m \geq N$, $n, m \in Q$, $j \in \{0, \ldots J\}$ so that $d_x(x, x_j) < \delta$

$$d_Y(f_n(x), f_m(x)) \leq \underbrace{d_Y(f_n(x), f_n(x_j))}_{\wedge\ \varepsilon} + \underbrace{d_Y(f_n(x_j), f_m(x_j))}_{\wedge\ \varepsilon} + \underbrace{d_Y(f_m(x_j), f_m(x))}_{\wedge\ \varepsilon} < 3\varepsilon.$$

Hence $(f_n)_{n \in \hat{Q}}$ is uniformly Cauchy on K, hence it converges uniformly on K.

(Y is complete). $\square$

Example: $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto z^2$ $\qquad \mathcal{F} = \{f^n, n \in \mathbb{N}\}$.

$\Rightarrow \mathcal{F}|_{\mathbb{D}}$ is normal and locally equicontinuous.

$\mathcal{F}|_{\mathbb{C} \setminus \bar{\mathbb{D}}}$ is normal but not locally equicontinuous. In fact any sequence in $\mathcal{F}|_{\mathbb{C} \setminus \bar{\mathbb{D}}}$ diverges uniformly from $\mathbb{C}$.

If $\mathcal{F}$ is considered as a family in $\mathrm{Hol}(\hat{\mathbb{C}}, \hat{\mathbb{C}})$, then $\mathcal{F}|_{\mathbb{D}}$ and $\mathcal{F}|_{\hat{\mathbb{C}} \setminus \mathbb{D}}$ are both normal and equicontinuous w.r.t. the spherical distance on $\hat{\mathbb{C}}$.

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Remark. Let $\mathcal{F} \subset \cancel{\mathcal{H}}\mathcal{O}(X,Y)$ be a family of holomorphic functions between Riemann surfaces (or more generally $\mathcal{F} \subset \mathcal{C}^0(X,Y)$ ...)

Then there is a maximal open subset $U \subset \mathcal{F}$ where $\mathcal{F}|_U$ is locally equicontinuous (or normal) (*)

Def.: We say that a family $\mathcal{F}$ is ~~locally~~ normal / locally equicontinuous at a point $x_0 \in X$ if there exist a open neighborhood $U \subset X$ of x_0 so that $\mathcal{F}|_U$ is normal / locally equicontinuous.

Notice that if $\mathcal{F}$ is normal / locally equicontinuous at x_0, and U is an open ~~set~~ given by the definition, then $\mathcal{F}$ is normal / loc equicont. at x $\forall x \in U$.

(*): Suppose Y compact (so that normal $\Leftrightarrow$ locally equicontinuous).

If $\mathcal{F}|_{U_\alpha}$ is normal $\forall \alpha$, then $\mathcal{F}|_U$ is normal in $U = \bigcup_\alpha U_\alpha$.

It is easy to see by equicontinuity: $\forall K \subset U$, compact, we can cover it by finitely many open sets V_j ~~that are~~ such that V_j, $\bar{V}_j$ is relatively compact in U_{α_j} for some α_j. Then the equicontinuity of $\mathcal{F}|_{V_j}$ plus the fact that V_j are in finite number gives the equicontinuity of $\mathcal{F}|_K$.